

LOCAL RINGS OF BOUNDED MODULE TYPE ARE ALMOST MAXIMAL VALUATION RINGS

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ABSTRACT. It is shown that every commutative local ring of bounded module type is an almost maximal valuation ring.

We say that an associative commutative unitary ring R is of bounded module type if there exists a positive integer n such that every finitely generated R -module is a direct sum of submodules generated by at most n elements. For instance, every Dedekind domain has bounded module type with bound $n = 2$. Warfield [9, Theorem 2] proved that every commutative local ring of bounded module type is a valuation ring. Moreover, Gill [5] and Lafon [7] showed independently that a valuation ring R is almost maximal if and only if every finitely generated R -module is a direct sum of cyclic modules. So every almost maximal valuation ring has bounded module type and Vámos proposed the following conjecture, in the Udine Conference on Abelian Groups and Modules, held in 1984:

“A local ring of bounded module type is an almost maximal valuation ring.”

P. Zanardo [10] and P. Vámos [8] first investigated this conjecture and proved it respectively for strongly discrete valuation domains and \mathbb{Q} -algebra valuation domains. More recently the author proved the following theorem (see [1, Corollary 9 and Theorem 10]).

Theorem 1. *Let R be a local ring of bounded module type. Suppose that R satisfies one of the following conditions:*

- (1) *There exists a nonmaximal prime ideal J such that R/J is almost maximal.*
- (2) *The maximal ideal of R is the union of all nonmaximal prime ideals.*

Then R is an almost maximal valuation ring.

From this theorem we deduce that it is enough to show the following proposition to prove Vámos’ conjecture : see [1, remark 1.1]. Recall that a valuation ring is *archimedean* if the maximal ideal is the only non-zero prime ideal.

Proposition 2. *Let R be an archimedean valuation ring for which there exists a positive integer n such that every finitely generated uniform module is generated by at most n elements. Then R is almost maximal.*

In the sequel we use the same terminology and the same notations as in [1] except for the symbol $A \subset B$ which means that A is a proper subset of B . We also use the terminology of [2]. Some results of [2] and the following lemmas will be useful to show proposition 2.

An R -module E is said to be *fp-injective* (or absolutely pure) if $\text{Ext}_R^1(F, E) = 0$, for every finitely presented R -module F . A ring R is called *self fp-injective* if it is fp-injective as R -module.

Lemma 3. *Let R be a coherent archimedean valuation ring, Y an injective R -module, X a pure submodule of Y and $Z = Y/X$. Then Z is injective.*

Proof. We have X fp-injective because it is a pure submodule of an injective module. So, since R is coherent, Z is also fp-injective by [3, Theorem 1.5]. Let J be an ideal of R and $f : J \rightarrow Z$ an homomorphism. By [2, corollary 36], J is countably generated. Then there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of J such that $J = \cup_{n \in \mathbb{N}} Ra_n$ and $a_n \in Ra_{n+1} \forall n \in \mathbb{N}$. Since Z is fp-injective, for every $n \in \mathbb{N}$ there exist $z_n \in Z$ such that $f(a_n) = a_n z_n$. It follows that $a_n(z_{n+1} - z_n) = 0$, $\forall n \in \mathbb{N}$. By induction on n we build a sequence $(y_n)_{n \in \mathbb{N}}$ of elements of Y such that $z_n = y_n + X$ and $a_n(y_{n+1} - y_n) = 0$, $\forall n \in \mathbb{N}$. Suppose y_0, \dots, y_n are built. Let $y'_{n+1} \in Y$ such that $z_{n+1} = y'_{n+1} + X$. Then $a_n(y'_{n+1} - y_n) \in X$. Since X is a pure submodule of Y there exists $x_{n+1} \in X$ such that $a_n(y'_{n+1} - y_n) = a_n x_{n+1}$. We put $y_{n+1} = y'_{n+1} - x_{n+1}$. The injectivity of Y implies that there exists $y \in Y$ such that $a_n(y - y_n) = 0$, $\forall n \in \mathbb{N}$. We set $z = y + X$. Then it is easy to check that $f(a) = az$ for every $a \in J$. \square

We say that a ring R is an *IF-ring* if it is coherent and self fp-injective. Equivalent conditions for a valuation ring to be an IF-ring are given in [2, Theorem 10].

In the four following lemmas R is a non-noetherian archimedean valuation ring and an IF-ring. In the sequel P is the maximal ideal of R .

Lemma 4. *Then:*

- (1) P is the only prime ideal of R .
- (2) P is not finitely generated.
- (3) P is faithful.

Proof. Since R is self fp-injective, $\forall a \in P$, $(0 : a) \neq 0$ by [1, Proposition 1(1)]. Hence P is the only prime ideal.

If P is finitely generated then all prime ideals of R are finitely generated. It follows that R is noetherian.

If P is not faithful then there exists $a \in R$ such that $P = (0 : a)$. We deduce that P is finitely generated because R is coherent, whence a contradiction. So P is faithful. \square

Lemma 5. *Let U be a non-zero uniserial R -module such that, for each $x \in U \setminus \{0\}$ $(0 : x)$ is not finitely generated. Then U is fp-injective if and only if U is faithful.*

Proof. Assume that U is faithful. Let $s \in R$ and $x \in U$ such that $(0 : s) \subseteq (0 : x)$. We will show that $x \in sU$. Since R is coherent $(0 : s)$ is principal, whence we have $(0 : s) \subset (0 : x)$. Hence $s(0 : x) \neq 0$. So, $\exists u \in U$ such that $(0 : u) \subset s(0 : x) \subseteq (0 : x)$ because U is faithful and uniserial. Hence $x \in Ru$ and there exists $r \in R$ such that $x = ru$. By [2, Lemma 2] $(0 : u) = r(0 : x)$. It follows that $r(0 : x) \subset s(0 : x)$. Therefore $r \in Rs$. We get that $x = sau$ for some $a \in R$.

Conversely, assume that U is fp-injective. Since P is faithful and the only prime ideal by lemma 4, we can apply [2, Proposition 27]. Hence U is faithful. \square

Lemma 6. *Let E be a non-zero fp-injective R -module. Then E contains a faithful uniserial pure submodule U . Moreover, if there exists $y \in E$ such that $(0 : y) = 0$ then we can choose $U = Ry$. Else, U is not finitely generated.*

Proof. If there exists $y \in E$ such that $(0 : y) = 0$, we set $U = Ry$. Since $U \simeq R$, U is fp-injective. So it is a pure submodule of E .

Suppose now that $(0 : x) \neq 0, \forall x \in E$. We put $I = (0 : x)$ for a non-zero element x of E and $J = (0 : I)$. If $0 \neq y \in E(R/I)$ and $(0 : y) \subseteq I$ then $(0 : y) = sI$ for some $s \in R \setminus J$ (see [4, Proposition IX.2.1]). Since P is faithful and the only prime ideal by lemma 4, then $E(R/I)$ is faithful by [2, Proposition 27]. So $\cap_{s \in R \setminus J} sI = 0$. On the other hand, R is countably cogenerated by [2, Corollary 35]. By [2, Lemma 30] there exists a countable family $(s_n)_{n \in \mathbb{N}}$ of elements of $R \setminus J$ such that $\cap_{n \in \mathbb{N}} s_n I = 0$ and $s_n \notin Rs_{n+1}$ for each $n \in \mathbb{N}$. We put $t_0 = 1, t_1 = s_1$. For each integer $n \geq 1$, let $t_{n+1} \in P$ such that $s_{n+1} = t_{n+1}s_n$. We get a family $(t_n)_{n \in \mathbb{N}}$ of elements of R such that $t_0 \dots t_n I \neq 0, \forall n \in \mathbb{N}$, and $\cap_{n \geq 0} t_0 \dots t_n I = 0$. By induction on n , we build a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of E such that $(0 : x_n) = t_0 \dots t_n I, \forall n \in \mathbb{N}$. We set $x_0 = x$. Since $t_{n+1}(0 : x_n) \neq 0$ we have $(0 : t_{n+1}) \subset (0 : x_n)$. So, the fp-injectivity of E implies that $\exists x_{n+1} \in E$ such that $x_n = t_{n+1}x_{n+1}$. By [2, Lemma 2] it follows that $(0 : x_{n+1}) = t_{n+1}(0 : x_n) = t_0 \dots t_{n+1}I$. Let U be the submodule of E generated by $\{x_n \mid n \geq 0\}$. Then U is uniserial and faithful. Suppose $\exists u \in U$ such that $(0 : u)$ is principal. Since R is an IF-ring there exists $s \in R$ such that $(0 : u) = (0 : s)$. The fp-injectivity of E implies that $u = se$ for some $e \in E$. It follows that $(0 : e) = s(0 : u) = 0$ by [2, Lemma 2]. We get a contradiction. Thus we can apply lemma 5. Hence U is fp-injective. So, it is a pure submodule of E . Then U is not finitely generated, else $U = Ry$ with $(0 : y) = 0$. \square

If N is a finitely generated R -module, $\mu(N)$ denotes its minimal number of generators. Then $\mu(N) = \dim_{R/P} N/PN$. If M is an R -module, $\mathcal{F}(M)$ is the family of its finitely generated submodules and we put $\nu(M) = \sup\{\mu(M') \mid M' \in \mathcal{F}(M)\}$.

Lemma 7. *Let E be a non-zero fp-injective module. If $\nu(E) \leq n$ then there exists a non-zero uniserial pure submodule U of E such that $\nu(E/U) \leq n - 1$.*

Proof. By lemma 6 there exists a uniserial pure submodule U . Let M be a submodule of E/U generated by $\{y_1, \dots, y_p\}$. We assume that $\mu(M) = p$. Let $x_1, \dots, x_p \in E$ such that $y_k = x_k + U$ and F be the submodule of E generated by x_1, \dots, x_p .

First we assume that U is finitely generated, so $U = Rx_0$, for some $x_0 \in E$. If $F \cap U = U$ then $U \subseteq F$ and U is a pure submodule of F . It follows that the following sequence is exact:

$$0 \rightarrow \frac{U}{PU} \rightarrow \frac{F}{PF} \rightarrow \frac{M}{PM} \rightarrow 0.$$

So, we have $\mu(M) = \mu(F) - \mu(U) = p - 1$. We get a contradiction since $\mu(M) = p$. Hence $F \cap U \neq U$. Let N be the submodule of E generated by x_0, x_1, \dots, x_p . Clearly $Rx_0 = N \cap U$. It follows that $N/(N \cap U) \simeq M$. Then $\mu(N) = p + 1$. Hence $p \leq n - 1$.

Now, suppose that U is not finitely generated. So, we may assume that $(0 : x) \neq 0$ for each $x \in E$. Since U is faithful, there exists $x_0 \in U$ such that $(0 : x_0) \subset \cap_{i=1}^{i=p} (0 : x_i)$. Clearly $F \cap U \neq U$. Let N be the submodule of E generated by x_0, x_1, \dots, x_p . Then Rx_0 is a pure submodule of N : see proof of [9, Theorem 2]. Clearly $Rx_0 = N \cap U$. As above we conclude that $p \leq n - 1$. \square

Proof of proposition 2. Let \mathcal{F} be the family of non-zero ideals A of R for which R/A is not maximal. If $\mathcal{F} \neq \emptyset$ we set $J = \cup_{A \in \mathcal{F}} A$. Then J is a prime ideal by [4, Lemma II.6.5]. Since R is archimedean, we have either $J = P$ or $\mathcal{F} = \emptyset$. So R is almost maximal if and only if there exists a non-zero proper ideal $I \neq P$ such

that R/I is maximal. Consequently, we may replace R by R/rR for some nonzero element r of P and assume that R is an IF-ring by [2, Theorem 11]. Let E be a non-zero injective indecomposable R -module. Since $\nu(E) \leq n$, by using lemma 7, there exists a pure-composition series of E of length $m \leq n$,

$$0 = E_0 \subset E_1 \subset \dots \subset E_m = E$$

such that E_k/E_{k-1} is uniserial and fp -injective, $\forall k, 1 \leq k \leq m$. By lemma 3, $U = E/E_{m-1}$ is injective. Since U is faithful by lemma 6 there exists $x \in U$ such that $\{0\} \subset (0 : x) \subset P$. We set $I = (0 : x)$. Let $V = \{y \in U \mid I \subseteq (0 : y)\}$. Then V is an injective module over R/I . Assume that there exists $y \in V$ such that $x = sy$ for some $s \in R$. By [2, Lemma 2] we get $I \subseteq (0 : y) = sI$. As in [4, p. 69] let $I^\# = \{r \in R \mid rI \subset I\}$. By [4, Lemma II.4.3] $I^\#$ is a prime ideal. So, $I^\# = P$ because P is the only prime ideal. The equality $sI = I$ implies that s is a unit. It follows that $V = Rx \simeq R/I$. So R/I is self-injective. By [6, Theorem 2.3] R/I is maximal. Hence R is almost maximal. \square

The proof of the following theorem is now complete.

Theorem 8. *Every commutative local ring of bounded module type is an almost maximal valuation ring.*

From proposition 2 we also deduce the following corollary.

Corollary 9. *Let R be an archimedean valuation domain, \tilde{R} a maximal immediate extension of R , Q and \tilde{Q} their respective fields of fractions. If $[\tilde{Q} : Q] < \infty$, then R is almost maximal.*

Proof. The conclusion holds since $\mu(M) \leq [\tilde{Q} : Q]$ for every finitely generated uniform module M by [11, Theorem 2.2]. \square

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